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On Convergence and Stability of the Generalized Noor Iterations for a General Class of Operators

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Abstract

In this paper, we establish some strong convergence and stability results of multistep iterative scheme for a general class of operators introduced by Bosede and Rhoades [5] in a Banach space. As corollaries, some convergence and stability results for the Noor, Ishikawa, Mann and Picard iterative schemes are also established. In the Banach space setting, our convergence results generalize and extend the results of Berinde [3], Bosede [4], Olaleru [16], Rafiq [21, 22] among others, while our stability results are extensions and generalizations of multitude of results in the literature, including the results of Berinde [1], Bosede and Rhoades [5], Imoru and Olatinwo [9] and Osilike [18].

Keywords: Strong convergence results, Stability results, multistep, Noor, Ishikawa, Mann and Picard iterative schemes.

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1 Introduction and Preliminary Definitions

Let (X, d) be a metric space and $T : X \to X$ be a selfmap of X. Assume that $F_T = p \in X : T_p = p$ is the set of fixed points of T. For $x_0 \in X$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n \geq 0. \tag{1.1}$$

is called the Picard iterative scheme.

We shall also need the following iterative schemes which appear in [13], [10], [14] and [28] respectively to establish our results.
Let E be a Banach space and $T : E \to E$ a self map of E. For $x_0 \in E$, the sequence $\{x_n\}_{n=1}^{\infty}$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the Mann iterative

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scheme [13].

If $\alpha_n = 1$ in (1.2), we have the Picard iterative scheme (1.1).

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called Ishikawa iterative scheme [10].

Observe that if $\beta_n = 0$ for each n, then the Ishikawa iterative scheme (1.3) reduces to the Mann iterative scheme (1.2).

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1) such that
$\sum_{n=0}^{\infty} \alpha_n = \infty$ is called Noor iterative (or three-step) scheme [14].
Also observe that if $\gamma_n = 0$ for each n, then the Noor iteration process (1.4) reduces to the Ishikawa iterative scheme (1.3).

For $x_0 \in E$, $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n^1 \\ y_n^i &= (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, i = 1, 2, ..., k-2, \\ y_n^{k-1} &= (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T x_n, \quad k \geq 2 \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}, i = 1, 2, ..., k-1$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called multistep iterative scheme [28].

Observe that the multistep iterative scheme (1.5) is a generalization of Noor, Ishikawa and the Mann iterative schemes. Infact, if $k = 3$ in (1.5), we have the Noor iterative scheme (1.4), if $k = 2$ in (1.5), we have the Ishikawa iteration (1.3) if $k = 2$ and $\beta_n^1 = 0$ in (1.5), we have the Mann iterative scheme (1.2).

The first important result on T-stable mappings was established by Ostrowski [20] for Picard iterative scheme defined by (1.1) using the Banach contraction condition defined as follows:

$$d(Tx, Ty) \leq ad(x, y), \tag{1.6}$$

for all $x, y \in X$ and $0 \leq a < 1$.
Berinde [1], also gave the following remarkable explaination on the stability of iteration procedures. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an iteration procedure involving the operator T

$$x_{n+1} = f(T, x_n), \tag{1.7}$$

$n = 0, 1, 2, ...$, where $x_0 \in X$ is the initial approximation and f is some function. For example, the Picard iteration (1.1) is obtained from (1.7) for $f(T, x_n) = Tx_n$, while the Mann iteration (1.2) is obtained for $f(T, x_n) = (1-\alpha_n)x_n + \alpha_n T x_n$, with $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in [0,1) and X a normed

linear space. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. When calculating $\{x_n\}_{n=0}^{\infty}$, then we cover the following steps:
1. We chose the initial approximation $x_0 \in X$;
2. Then we compute $x_1 = f(T, x_0)$, but due to various errors (rounding errors, numerical approximations of functions, derivatives or integrals), we do not get the exact value of x_1, but a different one u_1, which is very close to x_1
3. Consequently, when computing $x_2 = f(T, x_1)$ we shall have actually $x_2 = f(T, u_1)$ and instead of the theoretical value x_2, we shall obtain a closed value and so on. In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iterative method, we get an approximant sequence $\{u_n\}_{n=0}^{\infty}$. We say the iteration method is stable if and only if for u_n closed enough to x_n, $\{u_n\}_{n=0}^{\infty}$ still converges to the fixed point p of T. Following this idea, Harder and Hicks [7] introduced the following concept of stability.

Definition 1.1. Let (X, d) be a metric space and $T : X \to X$ a self map, $x_0 \in X$ and the iteration procedure defined by (1.7) such that the generated sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{u_n\}_{n=0}^{\infty}$ be arbitrary sequence in X, and set $\epsilon_n = d(u_{n+1}, f(T, u_n))$, for $n \geq 0$. We say the iteration procedure (1.7) is $T-$stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $lim_{n\to\infty} u_n = p$.

Several generalizations of the Banach fixed point theorem have been proved to date, (for example see [6], [12] and [29]). One of the most commonly studied generalization hitherto is the one proved by Zamfirescu [29] in 1972, which is stated as thus:
Theorem 1.2. Let X be a complete metric space and $T : X \to X$ a Zamfirescu operator satisfying

$$d(Tx, Ty) \leq h \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \tag{1.8}$$

where $0 \leq h < 1$. Then, T has a unique fixed point and the Picard iteration (1.1) converges to p for any $x_0 \in X$.
Observe that in a Banach space setting, condition (1.8) implies

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\|, \tag{1.9}$$

where $0 \leq \delta < 1$ and $\delta = \max\{h, \frac{h}{2-h}\}$, for details of proof see[3].

Several papers have been written on the Zamfirescu operators (1.9), for example (see [3],[8], [22], [28]). The most commonly used methods of approximating the fixed points of the Zamfirescu operators are Picard, Mann [13], Ishikawa [10], Noor [14] and multistep [28] iterative schemes. Rhoades [23,24] used the Zamfirescu contraction condition (1.9) to obtan some convergence results for Mann and Ishikawa iterative schemes in a uniformly Banach space. Berinde [3] extended the results of the author [23,24] to arbitrary Banach space for the same fixed point iteration procedures. Rafiq [22], proved the convergence of Noor iterative scheme using the Zamfirescu operators defined by (1.9).

Some of the notable authors whose contributions are of paramont importance in the study of stabilty of the fixed point iterative schemes are: Ostrowski [20], Harder and Hicks [8], Rhoades [26, 27], Osilike [18], Osilike and Udomene [19], Jachymski [11] and Berinde [1]. The authors [8], [26, 27] and [18] used the method of summability theory of infinite matrices to establish some stability results using various contractive definitions. For example, Harder and Hick [8] proved several stability results under various contractive conditions including the Zamfirescu operators using definition (1.1). Rhoades [27], extended the results of [8] using the following contractive definition:
For $x, y \in X$, there exists $a \in [0, 1)$, such that

$$d(Tx, Ty) \leq a \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], d(x, Ty), d(y, Tx)\}, \tag{1.10}$$

Osilike [18] proved several stability results which are generalizations and extensions of most of the results of Rhoades [26, 27] using the following contractive definition: for each $x, y \in X$, there exist

$a \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq ad(x, y) + Ld(x, Tx). \tag{1.11}$$

Moreover, Osilike and Udomene [19] established a shorter method of proof in obtaining stability results for various iterative schemes using condition (1.9). Berinde [1] further established a shorter method of proof than that of Osilike and Udomene [19] using the same contractive definitions as in Harder and Hicks [8].

In 2003, Imoru and Olatinwo [9] proved some stability results and generalized some known results in the literature including the results of [1, 8, 18, 19, 20, 26, 27] using the follwing general contractive definition:
for each $x, y \in X$, there exist $a \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq ad(x, y) + \varphi(d(x, Tx)). \tag{1.12}$$

In a more recent paper, Bosede and Rhoades [5], made an obvious assumption implied by (1.11) and one which renders all generalizations of the form (1.12) pointless. That is if $x = p$ (is a fixed point), then (1.11) becomes

$$d(p, Ty) \leq ad(p, y). \tag{1.13}$$

for some $0 \leq a < 1$ and all $x, y \in X$.
The authors [5], established some stability results for Picard and Mann iterative schemes for a general class of operators defined by (1.13).

Unfortunately, we do not support the claim that all generalizations of the form (1.12) and (1.11) are pointless. Several generalizations can occur in the form of (1.12), (1.11) and even (1.9). [For example, see Bosede [6] for details]. This paper is saying that if $x = p$, then (1.13) is more general than (1.11), (1.12) and several others. Also, if by replacing L in (1.11) with more complicated expressions, the process of "generalizing" (1.11) could continue ad infinitum. Finally, the condition "$\varphi(0) = 0$" usually imposed by Imoru and Olatinwo [9] in the contractive definition (1.12) is no longer necessary in the contractive condition (1.13). However, Bosede [4] also proved strong convergence of Noor iterative process for this general class of functions.

In this paper, we use the general class of operators defined by (1.13) to establish some strong convergence and stability results for the multistep iterative scheme (1.5). As corollaries, some strong convergence and stability results are obtained for Noor, Ishikawa, Mann, and Picard iterative schemes for this general class of operators (1.13). Our convergence results generalize and extend the results of Berinde [3], Bosede [4], Olaleru [16], Rafiq [21, 22] among others, while our stability results are extensions and generalizations of a multitude of results in the literature, including the results of Bosede and Rhoades [5], Imoru and Olatinwo [9], Osilike [18] and Berinde [1].

We shall need the following Lemma which appear in [1], to prove our results.
Lemma 1.3 [1]: Let δ be a real number satisfying $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $lim_{n\to\infty}\epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \epsilon_n$, n=0,1,2,..., we have $lim_{n\to\infty}u_n = 0$.

2 Main Result

2.1. Some Strong Convergence Results in Banach Spaces

Theorem 2.1.1. Let $(E, ||.||)$ be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point p satisfying the condition

$$\|p - Ty\| \quad \leq \quad a\|p - y\|, \tag{2.1}$$

for each $y \in E$ and $0 \leq a < 1$. For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the multistep iterative scheme defined by (1.5) converging to p (that is $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty}$ are real sequences in $[0, 1)$ for $i = 1, 2, 3, ..., k-1$. Then the multistep iterative scheme converges strongly to p.

Proof:
In view of (1.13) and (1.5), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n)x_n + \alpha_n Ty_n^1 - (1 - \alpha_n + \alpha_n)p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Ty_n^1 - p\|. \end{aligned} \tag{2.2}$$

Using (2.1), with $y = y_n^1$, gives

$$\|Ty_n^1 - p\| \quad \leq \quad a\|y_n^1 - p\|. \tag{2.3}$$

Substituting (2.3) in (2.2), we have

$$\|x_{n+1} - p\| \quad \leq \quad (1 - \alpha_n)\|x_n - p\| + a\alpha_n\|y_n^1 - p\|. \tag{2.4}$$

We note that $\beta_n^i \in [0, 1)$, for $n = 1, 2, ...$ and $1 \leq i \leq k - 1$.

$$\begin{aligned} \|y_n^1 - p\| &= \|(1 - \beta_n^1)x_n + \beta_n^1 Ty_n^2 - (1 - \beta_n^1 + \beta_n^1)p\| \\ &\leq (1 - \beta_n^1)\|x_n - p\| + \beta_n^1\|Ty_n^2 - p\| \\ &\leq (1 - \beta_n^1)\|x_n - p\| + a\beta_n^1\|y_n^2 - p\| \\ &\leq (1 - \beta_n^1)\|x_n - p\| + a\beta_n^1[(1 - \beta_n^2)\|x_n - p\| + \beta_n^2\|Ty_n^3 - p\|] \\ &\leq (1 - \beta_n^1)\|x_n - p\| + a\beta_n^1(1 - \beta_n^2)\|x_n - p\| + a^2\beta_n^1\beta_n^2\|y_n^3 - p\| \\ &\leq (1 - \beta_n^1)\|x_n - p\| + a\beta_n^1(1 - \beta_n^2)\|x_n - p\| \\ &\quad + a^2\beta_n^1\beta_n^2[(1 - \beta_n^3)\|x_n - p\| + \beta_n^3\|Ty_n^4 - p\| \\ &\leq (1 - \beta_n^1)\|x_n - p\| + a\beta_n^1(1 - \beta_n^2)\|x_n - p\| \\ &\quad + a^2\beta_n^1\beta_n^2(1 - \beta_n^3)\|x_n - p\| + a^3\beta_n^1\beta_n^2\beta_n^3\|y_n^4 - p\|] \\ &\leq (1 - \beta_n^1)\|x_n - p\| + a\beta_n^1(1 - \beta_n^2)\|x_n - p\| \\ &\quad + a^2\beta_n^1\beta_n^2(1 - \beta_n^3)\|x_n - p\| + ... + a^{k-2}\beta_n^1\beta_n^2\beta_n^3..\beta_n^{k-2}\|y_n^{k-1} - p\| \\ &\leq (1 - \beta_n^1)\|x_n - p\| + a\beta_n^1(1 - \beta_n^2)\|x_n - p\| \\ &\quad + a^2\beta_n^1\beta_n^2(1 - \beta_n^3)\|x_n - p\| + ... \\ &\quad + a^{k-2}\beta_n^1\beta_n^2\beta_n^3..\beta_n^{k-2}[(1 - \beta_n^{k-1})\|x_n - p\| + a\beta_n^{k-1}\|x_n - p\|] \\ &\leq [1 - (1 - a)\beta_n^1 - (1 - a)a\beta_n^1\beta_n^2 - (1 - a)a^2\beta_n^1\beta_n^2\beta_n^3 \\ &\quad -..... - (1 - a)a^{k-2}\beta_n^1\beta_n^2\beta_n^3..\beta_n^{k-1}]]\|x_n - p\| \\ &\leq [1 - (1 - a)\beta_n^1]\|x_n - p\|. \end{aligned} \tag{2.5}$$

Substituting (2.5) in (2.4), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n)\|x_n - p\| + a\alpha_n[1 - (1 - a)\beta_n^1]\|x_n - p\| \\ &= [1 - (1 - a)\alpha_n]\|x_n - p\|. \end{aligned} \tag{2.6}$$

By (2.6), we inductively obtain

$$\|x_{n+1} - p\| \quad \leq \quad \prod_{m=0}^{n}[1 - (1 - a)\alpha_m]\|x_0 - p\|.$$

Using the fact that $0 \leq a < 1$, $\alpha_m, \beta_n^i \in [0,1)\ for\ i = 1,2,...,k-1$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, it result that $\lim_{n\to\infty} \prod_{m=0}^{n}[1-(1-a)\alpha_m] = 0$.
Hence, $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.
That is $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. This ends the proof.

Remark 2.1.2. The class of maps satisfying (1.13) was newly introduced by Bosede and Rhoades [5] because of their simplicity and the fact that they generalize most single quasi-contractive maps if $x = p$. For example, the contractive maps (1.9), (1.10), (1.11) and (1.12) reduces to (1.13) if $x = p$ (fixed point). Hence the classes of maps defined by (1.13) is a generalization of those large classes of maps in (1.9), (1.10), (1.11) and (1.12) respectively.

Theorem 2.1.1. leads to the following corollaries:

Corollary 2.1.3. [4] Let $(E, ||.||)$ be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point p satisfying the condition

$$\|p - Ty\| \quad \leq \quad a\|p - y\|, \tag{2.7}$$

for each $y \in E$ and $0 \leq a < 1$. For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Noor iterative scheme defined by (1.4) converging to p (that is $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1)$. Then the Noor iterative scheme converges to p.

Remark 2.1.4. Corollary 2.1.3 generalizes the theorem of Rafiq [22] in the sense that if $x = p$, then the Zamfirescu operator used in Rafiq [21] reduces to the general class of map used in Corollary 2.1.3.

Corollary 2.1.5. Let $(E, ||.||)$ be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point p satisfying the condition

$$\|p - Ty\| \quad \leq \quad a\|p - y\|, \tag{2.8}$$

for each $y \in E$ and $0 \leq a < 1$. For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative scheme defined by (1.3) converging to p (that is $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1)$. Then the Ishikawa iterative scheme converges to p.

Corollary 2.1.6. Let $(E, ||.||)$ be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point p satisfying the condition

$$\|p - Ty\| \quad \leq \quad a\|p - y\|, \tag{2.9}$$

for each $y \in E$ and $0 \leq a < 1$. For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Mann iterative scheme defined by (1.2) converging to p (that is $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0, 1)$. Then the Mann iterative scheme converges to p.

Remark 2.1.7
(i) If $x = p$, the Zamfirescu condition given by

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\|,$$

where $0 \leq \delta < 1$, reduces to the general class of operator considered in corollary 2.1.5. Hence the convergence theorem of Berinde [1] is obtained in corollary 2.1.5.
(ii) The Chatterjea's and Kannan's contractive conditions (Z_2) and(Z_3) are both included in the class of Zamfirescu operators and so their convergence theorems for the Ishikawa iteration process (1.3)

are obtained in corollary 2.1.5.

2.2. Some Stability Results for Multistep Iterative Schemes in Banach Spaces

In this section, some stability results for the multistep iteratives defined by (1.5) are established for a general class of operators defined introduced by Bosede and Rhoades [5]. The stabilities of Noor, Ishikawa, Mann and Picard iterative schemes follow as corollaries. The theorem is stated thus:

Theorem 2.2.1. Let E be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point p and satisfying the condition

$$\|p - Ty\| \leq a\|p - y\|, \tag{2.10}$$

for each $y \in E$ and $0 \leq a < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the multistep iterative scheme defined by (1.5) converging to p with $0 < \alpha < \alpha_n$, $0 < \beta^i < \beta_n^i$ for $i = 1, 2, ..., k-1$ and all n. Then the multistep iterative scheme defined by (1.5) is T-stable.

Proof:
Let $\{z_n\}_{n=0}^{\infty}$, $\{u_n^i\}_{n=0}^{\infty}$, for $i = 1, 2, ...k-1$ be real sequences in E.
Let $\epsilon_n = \|z_{n+1} - (1-\alpha_n)z_n - \alpha_n T u_n^1\|$, $n = 0, 1, 2, ...$,
where
$u_n^i = (1-\beta_n^i)z_n + \beta_n^i T u_n^{i+1}$, $i = 1, 2, ..., k-2$
$u_n^{k-1} = (1-\beta_n^{k-1})z_n + \beta_n^{k-1} T z_n$, $i = 1, 2, ..., k-2$ and let $lim_{n\to\infty}\epsilon_n = 0$.
Then we shall prove that $lim_{n\to\infty} z_n = p$ using the general class of mappings satisfying condition (1.13).
That is,

$$\begin{aligned} \|z_{n+1} - p\| &\leq \|z_{n+1} - (1-\alpha_n)z_n - \alpha_n T u_n^1\| \\ &\quad + \|(1-\alpha_n)z_n + \alpha_n T u_n^1 - (1-\alpha_n+\alpha_n)p\| \\ &\leq \epsilon_n + (1-\alpha_n)\|z_n - p\| + \alpha_n\|T u_n^1 - p\|. \end{aligned} \tag{2.11}$$

Using condition (1.13), with $y = u_n^1$, we have

$$\|T u_n^1 - p\| \leq a\|u_n^1 - p\|. \tag{2.12}$$

Substituting (2.12) in (2.11), we have

$$\|z_{n+1} - p\| \leq \epsilon_n + (1-\alpha_n)\|z_n - p\| + a\alpha_n\|u_n^1 - p\|. \tag{2.13}$$

$$
\begin{aligned}
\|u_n^1 - p\| &= \|(1-\beta_n^1)z_n + \beta_n^1 T u_n^2 - (1-\beta_n^1+\beta_n^1)p\| \\
&\leq (1-\beta_n^1)\|z_n - p\| + \beta_n^1\|Tu_n^2 - p\| \\
&\leq (1-\beta_n^1)\|z_n - p\| + a\beta_n^1\|u_n^2 - p\| \\
&\leq (1-\beta_n^1)\|z_n - p\| + a\beta_n^1[(1-\beta_n^2)\|z_n - p\| + \beta_n^2\|Tu_n^3 - p\|] \\
&\leq (1-\beta_n^1)\|z_n - p\| + a\beta_n^1(1-\beta_n^2)\|z_n - p\| + a^2\beta_n^1\beta_n^2\|u_n^3 - p\| \\
&\leq (1-\beta_n^1)\|z_n - p\| + a\beta_n^1(1-\beta_n^2)\|z_n - p\| \\
&\quad + a^2\beta_n^1\beta_n^2[(1-\beta_n^3)\|z_n - p\| + \beta_n^3\|Tu_n^4 - p\| \\
&\leq (1-\beta_n^1)\|z_n - p\| + a\beta_n^1(1-\beta_n^2)\|z_n - p\| \\
&\quad + a^2\beta_n^1\beta_n^2(1-\beta_n^3)\|z_n - p\| + a^3\beta_n^1\beta_n^2\beta_n^3\|u_n^4 - p\|] \\
&\leq (1-\beta_n^1)\|z_n - p\| + a\beta_n^1(1-\beta_n^2)\|z_n - p\| \\
&\quad + a^2\beta_n^1\beta_n^2(1-\beta_n^3)\|z_n - p\| + \ldots + a^{k-2}\beta_n^1\beta_n^2\beta_n^3..\beta_n^{k-2}\|u_n^{k-1} - p\| \\
&\leq (1-\beta_n^1)\|z_n - p\| + a\beta_n^1(1-\beta_n^2)\|z_n - p\| \\
&\quad + a^2\beta_n^1\beta_n^2(1-\beta_n^3)\|z_n - p\| + \ldots \\
&\quad + a^{k-2}\beta_n^1\beta_n^2\beta_n^3..\beta_n^{k-2}[(1-\beta_n^{k-1})\|z_n - p\| + a\beta_n^{k-1}\|z_n - p\|] \\
&\leq [1-(1-a)\beta_n^1 - (1-a)a\beta_n^1\beta_n^2 - (1-a)a^2\beta_n^1\beta_n^2\beta_n^3 \\
&\quad -\ldots.. - (1-a)a^{k-2}\beta_n^1\beta_n^2\beta_n^3..\beta_n^{k-1}]\|z_n - p\| \\
&\leq [1-(1-a)\beta_n^1]\|z_n - p\|. \qquad (2.14)
\end{aligned}
$$

Substituting (2.14) in (2.13), we have

$$
\begin{aligned}
\|z_{n+1} - p\| &\leq \epsilon_n + (1-\alpha_n)\|z_n - p\| + a\alpha_n[1-(1-a)\beta_n^1]\|z_n - p\| \\
&= [1-\alpha_n + a\alpha_n - a\alpha_n\beta_n^1 + a^2\alpha_n\beta_n^1]\|z_n - p\| + \epsilon_n
\end{aligned}
$$

since $0 < \alpha < \alpha_n$, then

$$
\|z_{n+1} - p\| \leq [1-(1-a)\alpha]\|z_n - p\| + \epsilon_n. \qquad (2.15)
$$

Using lemma 1.3 in (2.15), we have

$$
lim_{n\to\infty} z_n = p.
$$

Conversely, let $lim_{n\to\infty} z_n = p$, we show that $lim_{n\to\infty}\epsilon_n = 0$ as follows:

$$
\begin{aligned}
\epsilon_n &= \|z_{n+1} - (1-\alpha_n)z_n - \alpha_n T u_n^1\| \\
&\leq \|z_{n+1} - p\| + \|(1-\alpha_n+\alpha_n)p - (1-\alpha_n)z_n - \alpha_n T u_n^1\| \\
&\leq \|z_{n+1} - p\| + (1-\alpha_n)\|z_n - p\| + a\alpha_n\|u_n^1 - p\|. \qquad (2.16)
\end{aligned}
$$

Substituting (2.14) in (2.16), we have

$$
\begin{aligned}
\epsilon_n &\leq \|z_{n+1} - p\| + (1-\alpha_n)\|z_n - p\| + a\alpha_n[1-(1-a)\alpha_n]\|z_n - p\| \\
&\leq \|z_{n+1} - p\| + [1-(1-a)\alpha]\|z_n - p\|. \qquad (2.17)
\end{aligned}
$$

Since $\lim_{n\to\infty}\|z_n - p\| = 0$ (by our assumption).
Hence, $lim_{n\to\infty}\epsilon_n = 0$.
Therefore the multistep iterative scheme (1.13) is T-stable.

Theorem 2.2.1 yields the following corollaries:
Corollary 2.2.2. Let E be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point and satisfying the condition

$$
\|p - Ty\| \leq a\|p - y\|,
$$

for each $y \in E$ and $0 \leq a < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iterative scheme defined by (1.4) converging to p with $0 < \alpha < \alpha_n$, $0 < \beta < \beta_n$, $0 < \gamma < \gamma_n$ for all n. Then the Noor iterative scheme defined by (1.4) is T-stable.

Remark 2.2.3. Corollary 2.2.2 is a generalization and extension of Theorem 3 of Berinde [1], Theorem 2 of Osilike [18], Theorem 2 and Theorem 5 of Osilike and Udomene [19], Theorem 2 of Rhoades [27], Theorem 3 of Harder and Hicks [8]. Some of the results are stated below and the proofs follow from Corollary 2.2.2 in view of the fact that the Picard, Mann and Ishikawa iteration are special cases of the Noor iterative scheme.

Corollary 2.2.4. Let E be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point and satisfying the condition

$$\|p - Ty\| \leq a\|p - y\|,$$

for each $y \in E$ and $0 \leq a < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative scheme defined by (1.3) converging to p with $0 < \alpha < \alpha_n$, $0 < \beta < \beta_n$ for all n. Then the Ishikawa iterative scheme defined by (1.3) is T-stable.

Corollary 2.2.5 (Theorem 2.2 of Bosede and Rhoades [5]). Let E be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point and satisfying the condition

$$\|p - Ty\| \leq a\|p - y\|,$$

for each $y \in E$ and $0 \leq a < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iterative scheme defined by (1.2) converging to p with $0 < \alpha < \alpha_n$ for all n. Then the Mann iterative scheme defined by (1.2) is T-stable.

Corollary 2.2.6. Let E be a Banach space, $T : E \to E$ be a selfmap of E with a fixed point and satisfying the condition

$$\|p - Ty\| \leq a\|p - y\|,$$

for each $y \in E$ and $0 \leq a < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iterative scheme defined by (1.1) converging to p for all n. Then the Picard iterative scheme defined by (1.1) is T-stable.

Remark 2.2.7. Corollaries 2.2.5 and 2.2.6 are the main results of Bosede and Rhoades [5] as well as an extension and generalization of Imoru and Olatinwo [9] and Berinde [1].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The first author wrote the entire paper while the second author proofread.

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